International Journal of Advanced and Applied Sciences, 4(5) 2017, Pages: 101-108

Contents lists available at Science-Gate

International Journal of Advanced and Applied Sciences

Journal homepage: http://www.science-gate.com/IJAAS.html

The $\int \Gamma^{2\lambda I}$ statistical convergence of real numbers over Musielak p-metric space

M. Kemal Ozdemir [1, *], Subramanian Nagarajan [2], Ayhan Esi [3]

[1]Department of Mathematics, Science and Arts Faculty, Inonu University, 44280, Malatya, Turkey
[2]Department of Mathematics, SASTRA University, Thanjavur-613 401, India
[3]Department of Mathematics, Science and Arts Faculty, Adiyaman University, 02040, Adiyaman, Turkey

A R T I C L E I N F O

Article history:
Received 24 November 2016
Received in revised form
27 March 2017
Accepted 27 March 2017

Keywords:
Analytic sequence
Musielak - modulus function
P- metric space
Ideal convergent
Fuzzy number
De la Vallee-Poussin mean

A B S T R A C T

In this paper, we introduce the concepts of $\int \Gamma^{2\lambda I}$ statistical convergence and strongly $\int \Gamma^{2\lambda I}$ of real numbers. It is also shown that $\Gamma^{2\lambda I}$ statistical convergence and strongly $\int \Gamma^{2\lambda I}$ are equivalent for analytic sequences of real numbers. We introduce certain new double sequence spaces of $\int \Gamma^{2\lambda}$ of fuzzy real numbers defined by $I -$ convergence using sequences of Musielak-Orlicz functions and also study some basic topological and algebraic properties of these spaces, investigate the inclusion relations between these spaces.

1. Introduction

Consider w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication. Throughout this article the space of regularly gai multiple sequence defined over a semi-normed space (X, q), semi-normed by q will be denoted by $\chi^{2R}_{mn}(q)$ and $\Lambda^{2R}_{mn}(q)$ For $X = \mathbb{C}$, the field of complex numbers, these spaces represent the corresponding scalar sequence spaces. Some initial works on double sequence spaces is found in Bromwich (2005). Later on, they were investigated by Hardy (1904), Moricz (1991), Moricz and Rhoades (1988), Basarir and Solancan (1999), Tripathy (2003), Turkmenoglu (1999), and many others. We procure the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{(x_{mn}) \in w^2 : \sup_{m,n\in N} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$$\mathcal{C}_p(t) := \left\{(x_{mn}) \in w^2 : p - \lim_{m,n\to\infty} |x_{mn} - ł|^{t_{mn}} = 1 \text{ for some } ł \in \mathbb{C}\right\},$$

$$\mathcal{C}_{0p}(t) := \left\{(x_{mn}) \in w^2 : p - \lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1\right\},$$

$$\mathcal{L}_u(t) := \left\{(x_{mn}) \in w^2 : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);$$

where, $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - \lim_{m,n\to\infty}$ denotes the limit in the Pringsheim (1900) sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}$; $\mathcal{M}_u(t)$, $\mathcal{C}_p(t)$, $\mathcal{C}_{0p}(t)$, $\mathcal{L}_u(t)$, $\mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u$, $\mathcal{C}_p$, $\mathcal{C}_{0p}$, $\mathcal{L}_u$, $\mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Çolak (2004, 2005) have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t)$, $\mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha -$, $\beta -$, $\gamma -$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Zeltser (2001) has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely (2003) and Tripathy (2003) have independently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Altay and Başar (2005) have defined the spaces $\mathcal{BS}$, $\mathcal{BS}(t)$, $\mathcal{CS}_p$, $\mathcal{CS}_{bp}$, $\mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u$, $\mathcal{M}_u(t)$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha -$ duals of the spaces $\mathcal{BS}$, $\mathcal{BV}$, $\mathcal{CS}_{bp}$ and the

* Corresponding Author.
Email Address: kozdemir73@gmail.com (M. K. Ozdemir)
https://doi.org/10.21833/ijaas.2017.05.018
2313-626X/

$\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Başar and Sever (2009) have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Subramanian and Misra (2011) have studied the space $\chi^2_M(p,q,u)$ of double sequences and gave some inclusion relations.

The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox (1986) as an extension of the definition of strongly Cesàro summable sequences. Connor (1988) further extended this definition to a definition of strong $A-$ summability with respect to a modulus where $A=(a_{n,k})$ is a nonnegative regular matrix and established some connections between strong $A-$ summability, strong $A-$ summability with respect to a modulus, and $A-$ statistical convergence. In the notion of convergence of double sequences was presented by Pringsheim (1900). Also, the four dimensional matrix transformation $(Ax)_{k,\ell}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{k\ell}^{mn}x_{mn}$ was studied extensively by Hamilton (1936).

We need the following inequality in the sequel of the paper. For $a,b\geq 0$ and $0<p<1$, we have

$$(a+b)^p \leq a^p+b^p.$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn}=\sum_{i,j=1}^{m,n} x_{ij}(m,n\in\mathbb{N})$.

A sequence $x=(x_{mn})$is said to be double analytic if $\sup_{mn}|x_{mn}|^{1/m+n}<\infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x=(x_{mn})$ is called double gai sequence if $\left((m+n)!\,|x_{mn}|\right)^{1/m+n}\to 0$ as $m,n\to\infty$. The double gai sequences will be denoted by χ^2. Let $\phi=\{\text{finite sequences}\}$.

Consider a double sequence $x=(x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]}=\sum_{i,j=0}^{m,n} x_{ij}\mathfrak{I}_{ij}$ for all $m,n\in\mathbb{N}$; where $\mathfrak{I}_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i,j\in\mathbb{N}$.

An FK-space (or a metric space) X is said to have AK property if $(\mathfrak{I}_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]}\to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x=(x_k)\to(x_{mn})(m,n\in\mathbb{N})$ are also continuous.

Let M and Φ are mutually complementary modulus functions. Then, we have:

(i) For all $u,y\geq 0$,
$uy\leq M(u)+\Phi(y)$, (Young's inequality)

(ii) For all $u\geq 0$,

$$u\eta(u)=M(u)+\Phi(\eta(u)).$$

(iii) For all $u\geq 0$, and $0<\lambda<1$,

$$M(\lambda u)\leq \lambda M(u)$$

Lindenstrauss and Tzafriri (1971) used the idea of Orlicz function to construct Orlicz sequence space,

$$\ell_M=\left\{x\in w:\sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right)<\infty,\quad \text{for some}\quad \rho>0\right\},$$

the space ℓ_M with the norm,

$$\|x\|=\inf\left\{\rho>0:\sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right)\leq 1\right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t)=t^p(1\leq p<\infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

A sequence $f=(f_{mn})$ of modulus function is called a Musielak-modulus function. A sequence $g=(g_{mn})$ defined by:

$$g_{mn}(v)=\sup\{|v|u-(f_{mn})(u):u\geq 0\},m,n=1,2,\ldots$$

is called the complementary function of a Musielak-modulus function f. For a given Musielak modulus function f, the Musielak-modulus sequence space t_f is defined as follows

$$t_f=\left\{x\in w^2:M_f(|x_{mn}|)^{1/m+n}\to 0\quad \text{as}\quad m,n\to\infty\right\},$$

where, M_f is a convex modular defined by:

$$M_f(x)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}(|x_{mn}|)^{1/m+n}, x=(x_{mn})\in t_f.$$

We consider t_f equipped with the Luxemburg metric:

$$d(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(\frac{|x_{mn}|^{1/m+n}}{mn}\right).$$

If X is a sequence space, we give the following definitions:

(i) $X'=$ the continuous dual of X;
(ii) $X^{\alpha}=\left\{a=(a_{mn}):\sum_{m,n=1}^{\infty}|a_{mn}x_{mn}|<\infty,\ \text{for each}\ x\in X\right\}$;
(iii) $X^{\beta}=\left\{a=(a_{mn}):\sum_{m,n=1}^{\infty}a_{mn}x_{mn}\ \text{is convegent, for each}\ x\in X\right\}$;
(iv) $X^{\gamma}=\left\{a=(a_{mn}):\sup_{mn\geq 1}\left|\sum_{m,n=1}^{M,N}a_{mn}x_{mn}\right|<\infty, \text{for each}\ x\in X\right\}$;
(v) Let X be an FK $-$space$\supset\phi$; then $X^f=\{f(\mathfrak{I}_{mn}):f\in X'\}$;
(vi) $X^{\delta}=\left\{a=(a_{mn}):\sup_{mn}|a_{mn}x_{mn}|^{\frac{1}{m}+n}<\infty,\ \text{for each}\ x\in X\right\}$;

X^{α}, X^{β}, X^{γ} are called $\alpha-$ (or Köthe-Toeplitz) dual of X, $\beta-$(orgeneralized-Köthe-

Toeplitz) dualof X, $\gamma -$dualofX,$\delta -$dualof Xrespectively. X^{α} is defined by Gupta and Kampthan. It is clear that $X^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\beta} \subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kızmaz (1981) as follows:

$$Z(\Delta) = \{x = (x_k) \in w: (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

Here, c, c_0 and ℓ_∞ denote the classes of convergent, null and bounded scalar valued single sequences respectively. The difference sequence space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \le p \le \infty$ by Altay and Başar (2005) and in the case $0 < p < 1$ by Altay and Başar (2005). The spaces $c(\Delta)$, $c_0(\Delta)$, $\ell_\infty(\Delta)$ and bv_p are Banach spaces normed by:

$$\|x\| = |x_1| + \sup_{k\ge 1}|\Delta x_k|$$

and

$$\|x\|_{bv_p} = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}}, (1 \le p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by:

$$Z(\Delta) = \{x = (x_{mn}) \in w^2: (\Delta x_{mn}) \in Z\}$$

where, $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$.

2. Some new integrated statistical convergence sequence spaces of fuzzy numbers

The main aim of this article is to introduce the following sequence spaces and examine topological and algebraic properties of the resulting sequence spaces. Let $p = (p_{mn})$ be a sequence of positive real numbers for all $m, n \in \mathbb{N}$, $f = (f_{mn})$ be a Musielak-modulus function, $\left(X, \left\|\left(d(x_1,0), d(x_2,0), \dots, d(x_{n-1},0)\right)\right\|_p\right)$ be a $p-$metric space, and (λ_{rs}^{-1}) be a sequence of non-zero scalars and $\mu_{mn}(X) = \bar{d}(t_{rs}, \bar{0})$ be a sequence of fuzzy numbers, we define the following sequence spaces as follows:

$$\left[\Gamma_{f\mu}^{2q}, \left\|\left(d(x_1,0), d(x_2,0), \dots, d(x_{n-1},0)\right)\right\|_p\right] = \lim_{r,s\to\infty} \left|\left\{m, n \in I_{rs}: \left[f_{mn}\left(\left\|\mu_{mn}(x), \begin{pmatrix} d(x_1,0), d(x_2,0), \dots, \\ d(x_{n-1},0)\end{pmatrix}\right\|_p\right)\right]^{q_{mn}} \ge \varepsilon\right\}\right| = 0,$$

uniformly in r, s.

In this case, we write $X_{mn} \to \bar{0}(\breve{S}_\lambda^F)$. The set of all statistically convergent sequences is denoted by $\breve{S}_\lambda^F$.

Let $X = (X_{mn})$ be a sequence of fuzzy numbers and $q = (q_{mn})$ be a sequence of strictly positive real numbers. Then the sequence $X = (X_{mn})$ is said to be strongly $\lambda -$convergent if there is a fuzzy number $\bar{0}$ such that,

$$\left[\Gamma_{f\mu}^{2q}, \left\|\left(d(x_1,0), d(x_2,0), \dots, d(x_{n-1},0)\right)\right\|_p\right] = \lim_{r,s\to\infty} \sum_{m\in I_{rs}} \sum_{n\in I_{rs}} \left[f_{mn}\left(\left\|\mu_{mn}(x), \left(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0)\right)\right\|_p\right)\right]^{q_{mn}} = 0,$$

uniformly in r, s.

In this case, we write $X_{mn} \to \bar{0}(\breve{w}_\lambda^F, q)$. The set of all strongly $\lambda -$convergent sequences is denoted by $(\breve{w}_\lambda^F, q)$.

Let $X = (X_{mn})$ be a sequence of fuzzy numbers. Then the sequence $X = (X_{mn})$ of fuzzy numbers is said to be double analytic if the set $\{t_{rs}: r, s \in \mathbb{N}\}$ of fuzzy numbers is double analytic and it is denoted by $\breve{\Lambda}^{2F}$. In this section we give some inclusion relations between strongly $\lambda -$convergence and $\lambda -$statistically convergence and show that they are equivalent for almost bounded sequences of fuzzy numbers. We also study the inclusion $\breve{S}^{2F} \subset \breve{S}^{2F}$ under certain restrictions on the sequence $\Lambda^2 = (\lambda_{rs})$.

Let $n \in \mathbb{N}$ and X be a real vector space of dimension w, where $n \le m$. A real valued function $d_p(x_1, \dots, x_n) = \| (d_1(x_1, 0), \dots, d_n(x_n, 0)) \|_p$ on X satisfying the following four conditions:

(i) $\| (d_1(x_1,0), \dots, d_n(x_n,0)) \|_p = 0$ if and only if $d_1(x_1,0), \dots, d_n(x_n,0)$ are linearly dependent,
(ii) $\| (d_1(x_1,0), \dots, d_n(x_n,0)) \|_p$ is invariant under permutation,
(iii) $\| \left(\alpha d_1(x_1,0), \dots, d_n(x_n,0)\right) \|_p = |\alpha|.$ $\| (d_1(x_1,0), \dots, d_n(x_n,0)) \|_p, \alpha \in \mathbb{R}$,
(iv) $d_p((x_1,y_1), (x_2,y_2), \dots, (x_n,y_n)) = (d_X(x_1,x_2,\dots,x_n)^p + d_Y(y_1,y_2,\dots,y_n)^p)^{1/p}$ for $1 \le p < \infty$; (or)
(v) $d((x_1,y_1), (x_2,y_2), \dots, (x_n,y_n)) := \sup\{d_X(x_1,x_2,\dots,x_n), d_Y(y_1,y_2,\dots,y_n)\}$,

for $x_1, x_2, \dots, x_n \in X$, $y_1, y_2, \dots, y_n \in Y$ is called the p product metric of the Cartesian product of n metric spaces.

Definition 2.1: Let X be a linear metric space. A function $\rho: X \to \mathbb{R}$ is called paranorm, if

(1) $\rho(x) \ge 0$, for all $x \in X$;
(2) $\rho(-x) = \rho(x)$, for all $x \in X$;
(3) $\rho(x + y) \le \rho(x) + \rho(y)$, for all $x, y \in X$;
(4) If (σ_{mn}) is a sequence of scalars with $\sigma_{mn} \to \sigma$ as $m, n \to \infty$ and (x_{mn}) is a sequence of vectors with

$\rho(x_{mn} - x) \to 0$ as $m, n \to \infty$, then $\rho(\sigma_{mn}x_{mn} - \sigma x) \to 0$ as $m, n \to \infty$.

A paranorm w for which $\rho(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, w) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (Wilansky, 1984).

The notion of deal convergence was introduced first by Kostyrko et al. (2000) as a generalization of statistical convergence which was further studied in toplogical spaces by Kumar (2007) and Kumar and Kumar (2008), and also more applications of ideals can be deals with various authors by Hazarika (2009, 2012a,b,c, 2013a,b, 2014a,b,c,d,), Hazarika and Savas (2011), Hazarika et al. (2014), Hazarika and Kumar (2014), Tripathy and Hazarika (2008, 2009, 2011).

Definition 2.2: A family $I \subset 2^Y$ of subsets of a non empty set Y is said to be an ideal in Y if

(1) $\phi \in I$
(2) $A, B \in I$ imply $A \cup B \in I$
(3) $A \in I, B \subset A$ imply $B \in I$.

While an admissible ideal I of Y further satisfies $\{x\} \in I$ for each $x \in Y$. Given $I \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a non trivial ideal in $\mathbb{N} \times \mathbb{N}$. A sequence $(x_{mn})_{m,n\in\mathbb{N}\times\mathbb{N}}$ in X is said to be $I-$convergent to $0 \in X$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \left\{m, n \in \mathbb{N} \times \mathbb{N}: \| (d_1(x_1, 0), \dots, d_n(x_n, 0)) - 0 \|_p \geq \varepsilon\right\}$ belongs to I.

Definition 2.3: A non-empty family of sets $F \subset 2^X$ is a filter on X if and only if

(1) $\phi \in F$
(2) for each $A, B \in F$, we have imply $A \cap B \in F$
(3) each $A \in F$ and each $A \subset B$, we have $B \in F$.

Definition 2.4: An ideal I is called non-trivial ideal if $I \neq \phi$ and $X \notin I$. Clearly $I \subset 2^X$ is a non-trivial ideal if and only if $F = F(I) = \{X - A: A \in I\}$ is a filter on X.

Definition 2.5: A non-trivial ideal $I \subset 2^X$ is called (i) admissible if and only if $\left\{\{x\}: x \in X\right\} \subset I$. (ii) maximal if there cannot exists any non-trivial ideal $J \neq I$ containing I as a subset.

If we take $I = I_f = \{A \subseteq \mathbb{N} \times \mathbb{N}: A$ is a finite subset$\}$. Then I_f is a non-trivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincides with the usual convergence. If we take $I = I_\delta = \{A \subseteq \mathbb{N} \times \mathbb{N}: \delta(A) = 0\}$ where $\delta(A)$ denote the asymptotic density of the set A. Then I_δ is a non-trivial admissible ideal of $\mathbb{N} \times \mathbb{N}$ and the corresponding convergence coincides with the statistical convergence.

Let D denote the set of all closed and bounded intervals $X = [x_1, x_2]$ on the real line $\mathbb{R} \times \mathbb{N}$. For $X, Y \in D$, we define $X \leq Y$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$, $d(X, Y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$, where $X = [x_1, x_2]$ and $Y = [y_1, y_2]$.

Then it can be easily seen that d defines a metric on D and (D, d) is a complete metric space. Also the relation "$\leq$" is a partial order on D. A fuzzy number X is a fuzzy subset of the real line $\mathbb{R} \times \mathbb{R}$ i.e. a mapping $X: R \to J(= [0,1])$ associating each real number t with its grade of membership $X(t)$.

Definition 2.6: A fuzzy number X is said to be (i) convex if $X(t) \geq X(s) \wedge X(r) = \min\{X(s), X(r)\}$, where $s < t < r$. (ii) normal if there exists $t_0 \in \mathbb{R} \times \mathbb{R}$ such that $X(t_0) = 1$. (iii) upper semi-continuous if for each $\varepsilon > 0, X^{-1}([0, a + \varepsilon])$ for all $a \in [0,1]$ is open in the usual topology of $\mathbb{R} \times \mathbb{R}$.

Let R(J) denote the set of all fuzzy numbers which are upper semicontinuous and have compact support, i.e. if $X \in \mathbb{R}(J) \times \mathbb{R}(J)$ the for any $\alpha \in [0,1], [X]^\alpha$ is compact, where $[X]^\alpha = \{t \in \mathbb{R} \times \mathbb{R}: X(t) \geq \alpha,\ \text{if}\alpha \in [0,1]\}$, $[X]^0 =$closure of $(\{t \in \mathbb{R} \times \mathbb{R}: X(t) > \alpha, \text{if}\alpha = 0\})$.

The set $\mathbb{R}$ of real numbers can be embedded $\mathbb{R}(J)$ if we define $\bar{r} \in \mathbb{R}(J) \times \mathbb{R}(J)$ by

$$\bar{r}(t) = \begin{cases} 1, & \text{if} t = r; \\ 0, & \text{if} t \neq r \end{cases}$$

the absolute value, $|X|$ of $X \in \mathbb{R}(J)$ is defined by

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\}, & \text{if} t \geq 0; \\ 0, & \text{if} t < 0 \end{cases}$$

define a mapping $\bar{d}: \mathbb{R}(J) \times \mathbb{R}(J) \to \mathbb{R}^+ \cup \{0\}$ by

$$\bar{d}(X, Y) = \sup_{0\leq\alpha\leq1} d([X]^\alpha, [Y]^\alpha).$$

It is known that $(\mathbb{R}(J), \bar{d})$ is a complete metric space.

Definition 2.7: A metric on $\mathbb{R}(J)$ is said to be translation invariant if $\bar{d}(X + Z, Y + Z) = \bar{d}(X, Y)$, for $X, Y, Z \in \mathbb{R}(J)$.

Definition 2.8: A sequence $X = (X_{mn})$ of fuzzy numbers is said to be (i) convergent to a fuzzy number X_0 if for every $\varepsilon > 0$, there exists a positive integer n_0 such that $\bar{d}(X_{mn}, X_0) < \varepsilon$ for all $n \geq n_0$. (ii) bounded if the set $\{X_{mn}: m, n \in \mathbb{N}\}$ of fuzzy numbers is bounded.

Definition 2.9: A sequence $X = (X_{mn})$ of fuzzy numbers is said to be (i) I-convergent to a fuzzy number X_0 if for each $\varepsilon > 0$ such that,

$$A = \left\{m, n \in \mathbb{N}: \bar{d}(X_{mn}, X_0) \geq \varepsilon\right\} \in I.$$

The fuzzy number X_0 is called $I-$limit of the sequence (X_{mn}) of fuzzy numbers and we write $I-\lim X_{mn} = X_0$. (ii) $I-$bounded if there exists $M > 0$ such that,

$$\left\{m, n \in \mathbb{N}: \bar{d}(X_{mn}, \bar{0}) > M\right\} \in I.$$

Definition 2.10: A sequence space E_F of fuzzy numbers is said to be (i) solid (or normal) if $(Y_{mn}) \in E_F$ whenever $(X_{mn}) \in E_F$ and $\bar{d}(Y_{mn}, \bar{0}) \leq \bar{d}(X_{mn}, \bar{0})$

for all $m, n \in \mathbb{N}$. (ii) symmetric if $(X_{mn}) \in E_F$ implies $(X_{\pi(mn)}) \in E_F$ where π is a permutation of $\mathbb{N} \times \mathbb{N}$.

Let $K = \{k_1 < k_2 < \cdots\} \subseteq \mathbb{N}$ and E be a sequence space. A K −step space of E is a sequence space,

$$\lambda_{mn}^{E} = \left\{\left(X_{m_p n_p}\right) \in w^2 : (m_p n_p) \in E\right\}.$$

A canonical preimage of a sequence $\left\{\left(x_{m_p n_p}\right)\right\} \in \lambda_K^E$ is a sequence $\{y_{mn}\} \in w^2$ defined as:

$$y_{mn} = \begin{cases} x_{mn}, & \text{if} m, n \in E \\ 0, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space λ_K^E is a set of canonical preimages of all elements in λ_K^E, i.e. y is in canonical preimage of λ_K^E if and only if y is canonical preimage of some $x \in \lambda_K^E$.

Definition 2.11: A sequence space E_F is said to be monotone if E_F contains the canonical pre-images of all its step spaces.

The following well-known inequality will be used throughout the article. Let $p = (p_{mn})$ be any sequence of positive real numbers with $0 \le p_{mn} \le \sup_{mn} p_{mn} = G$, $D = \max\{1, 2G - 1\}$ then

$$|a_{mn} + b_{mn}|^{p_{mn}} \le D(|a_{mn}|^{p_{mn}} + |b_{mn}|^{p_{mn}})$$

for all $m, n \in \mathbb{N}$ and $a_{mn}, b_{mn} \in \mathbb{C}$.

Also $|a_{mn}|^{p_{mn}} \le \max\{1, |a|^G\}$ for all $a \in \mathbb{C}$.

First we procure some known results; those will help in establishing the results of this article.

Lemma 2.12: A sequence space E_F is normal implies E_F is monotone. (For the crisp set case, one may refer to Kamthan and Gupta (1981)).

Lemma 2.13: based on Lemma 5.1 of (Kostyrko et al., 2000) If $I \subset 2^{\mathbb{N}}$ is a maximal ideal, then for each $A \subset \mathbb{N}$ we have either $A \in I$ or $\mathbb{N} - A \in I$.

Definition 2.14: A sequence $X = (X_{mn})$ of fuzzy numbers is a function X from the set $\mathbb{N} \times \mathbb{N}$ of natural numbers into $L(\mathbb{R}) \times L(\mathbb{R})$. The fuzzy number X_{mn} denotes the value of the function $m, n \in \mathbb{N}$.

We denote W^{2F} denotes the set of all sequences $X = (X_{mn})$ of fuzzy numbers.

Definition 2.15: A sequence $X = (X_{mn})$ of fuzzy numbers is said to be analytic if the set $\{X_{mn} : m, n \in \mathbb{N}\}$ of fuzzy numbers is analytic.

The notion of statistical convergence for a sequence of complex numbers was introduced by Fridy (1985) and many others. Over the years and under different names statistical convergence has been discussed in the different theories such as the theory of Fourier analysis, ergodic theory and number theory. Later on, it was further investigated from the sequence space point of view and linked with summability theory by Fridy (1985), Salat (1980), Connor (1988), and many others. This concept extends the idea to apply to sequences of fuzzy numbers with Kwon and Shim (2001), Et et al. (2005), Nuray and Savas (1995), and many others.

Definition 2.16: The sequence $X = (X_{mn})$ of fuzzy numbers is said to be almost convergent to a fuzzy number $\bar{0}$ if $\lim_{m,n\to\infty} d\left(t_{pm,qn}(X), \bar{0}\right) = 0$ uniformly in m, n, where $t_{pm,qn}(X) = \frac{1}{(m+1)(n+1)} \sum_{i=0}^{p} \sum_{j=0}^{q} \left(X_{i+m,j+n}\right)^{1/((i+m)+(j+n))}$.

This means that for every $\varepsilon > 0$, there exists a $p_0 q_0 \in \mathbb{N}$ such that $d\left(t_{pm,qn}(X), \bar{0}\right) < \varepsilon$, whenever $p, q \ge p_0 q_0$ and for all m, n.

Definition 2.17: A sequence $X = (X_{mn})$ of fuzzy numbers is said to be statistically convergent to a fuzzy number $\bar{0}$ if for every $\varepsilon > 0$,

$$\lim_{\frac{1}{rs}} \left|\left\{m \le r, n \le s : d\left(X_{mn}^{1/m+n}, \bar{0}\right) \ge \varepsilon\right\}\right| = 0.$$

The set of all statistically convergent sequences of fuzzy numbers is denoted by S^{2F}.

We note that if a sequence $X = (X_{mn})$ of fuzzy numbers converges to a fuzzy number $\bar{0}$, then it is statistically converges to $\bar{0}$. But the converse statement is not necessarily valid.

Let $\mu = (\lambda_{rs})$ be a non-decreasing sequence of positive real numbers tending to infinity and $\lambda_{11} = 1$ and $\lambda_{r+1,s+1} \le \lambda_{rs} + 1$, for all $r, s \in \mathbb{N}$.

The generalized de la Vallee-Poussin mean is defined by:

$$t_{rs}(x) = \frac{1}{\lambda_{rs}} \sum_{p \in I_r} \sum_{q \in I_s} |x_{mn}|^{1/m+n}$$

where, $I_{rs} = [r, s - \lambda_{rs} + 1, rs]$. A sequence $x = (x_{mn})$ of complex numbers is said to be (V, λ) − summable to a number if $t_{rs}(x) \to L$ as $r, s \to \infty$.

3. Main results

Theorem 3.1: If $\Gamma^2(X) \in \breve{S}_{\lambda}^{2F}$ and $c \in \mathbb{R}$, then

(a) $\breve{S}_{\lambda}^{2F} - \lim c\Gamma^2(X) = c\breve{S}_{\lambda}^{2F} - \lim \Gamma^2(X)$

(b) $\breve{S}_{\lambda}^{2F} - \lim \Gamma^2(X + Y) = \breve{S}_{\lambda}^{2F} - \lim \Gamma^2(X) + \breve{S}_{\lambda}^{2F} - \lim \Gamma^2(Y)$

Proof (a): Let $\Gamma^2(X) \in \breve{S}_{\lambda}^{2F}$ so that $\breve{S}_{\lambda}^{2F} - \lim \Gamma^2(X) = \bar{0}$, $c \in \mathbb{R}$ and $\varepsilon > 0$. Then the inequality,

$$\left|\left\{m, n \in I_{rs} : \left[f_{mn}\left(\left\|\mu_{mn}(cx), \left(d(x_1, 0), d(x_2, 0), \ldots, d(x_{n-1}, 0)\right)\right\|_p\right)\right]^{q_{mn}} \ge \varepsilon\right\}\right| \le \left|\left\{m, n \in I_{rs} : \left[f_{mn}\left(\left\|\mu_{mn}(x), \left(d(x_1, 0), d(x_2, 0), \cdots, d(x_{n-1}, 0)\right)\right\|_p\right)\right]^{q_{mn}} \ge \frac{\varepsilon}{|c|}\right\}\right|,$$

for all $r, s \in \mathbb{N}$.

Proof (b): Suppose that $\Gamma^2(X),\Gamma^2(Y)\in\breve{S}_\lambda^{2F}$ so that $\breve{S}_\lambda^{2F}-\lim\Gamma^2(X)=\bar{0}$ and $\breve{S}_\lambda^{2F}-\lim\Gamma^2(Y)=\bar{0}$. By Minkowski's inequality, we get,

$$\left[f_{mn}\left(\left\|\mu_{mn}(x+y),(d(x_1,0),d(x_2,0),\dots,d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}} \leq \left[f_{mn}\left(\left\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\dots,d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}} + \left[f_{mn}\left(\left\|\mu_{mn}(y),(d(x_1,0),d(x_2,0),\dots,d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}}.$$

Therefore given $\varepsilon>0$, for all $r,s\in\mathbb{N}$, we have,

$$\left|\left\{m,n\in I_{rs}:\left[f_{mn}\left(\left\|\mu_{mn}(x+y),(d(x_1,0),d(x_2,0),\dots,d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}}\geq\varepsilon\right\}\right| \leq \left|\left\{m,n\in I_{rs}:\left[f_{mn}\left(\left\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}}\geq\frac{\varepsilon}{2}\right\}\right| + \left|\left\{m,n\in I_{rs}:\left[f_{mn}\left(\left\|\mu_{mn}(y),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}}\geq\frac{\varepsilon}{2}\right\}\right|.$$

This completes the proof.

The following theorem shows that $\lambda-$ statistical convergence and strongly $\lambda-$ convergence are equivalent for double analytic sequences of fuzzy numbers.

Theorem 3.2: Let the sequence $\mu=(\mu_{mn})$ be double analytic and $\Gamma^2(X)$ be a sequence of fuzzy numbers. Then

(a) $\Gamma^2(X)\to\bar{0}(\breve{w}_\lambda^{2F},\mu)$ implies $\Gamma^2(X)\to\bar{0}(\breve{S}_\lambda^{2F},\mu)$.
(b) $\Lambda^2(X)\to\bar{0}(\breve{S}_\lambda^{2F},\mu)$ imply $\Lambda^2(X)\to\bar{0}(\breve{w}_\lambda^{2F},\mu)$.
(c) $\breve{S}_\lambda^{2F}\cap\ \Lambda_\lambda^{2F}=(\breve{w}_\lambda^{2F},\mu)\cap\ \Lambda_\lambda^{2F}$.

Proof (a): Let $\varepsilon>0$ and $\Gamma^2(X)\to\bar{0}(\breve{w}_\lambda^{2F},\mu)$ for all $r,s\in\mathbb{N}$, we have,

$$\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left[f_{mn}\left(\left\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\dots,d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}} \geq \sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left[f_{mn}\left(\left\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\dots,d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}} \geq \varepsilon$$

$$\left[f_{mn}\left(\left\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\dots,d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}} \geq \left|\left\{m,n\in I_{rs}:\left[f_{mn}\left(\left\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\dots,d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}}\geq\varepsilon\right\}\right|\cdot\min(\varepsilon^h,\varepsilon^H).$$

Hence $\Gamma^2(X)\in\breve{S}_\lambda^{2F}$.

Proof (b): Suppose that $\Gamma^2(X)\in\breve{S}_{2\lambda}^{2F}\cap\ \Lambda^{2F}$. Since $\Gamma^2(X)\in\Lambda^{2F}$, we write,

$$\left[f_{mn}\left(\left\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\dots,d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}} \leq T,$$

for all $r,s\in\mathbb{N}$, let

$$G_{rs} = \left|\left\{m,n\in I_{rs}:\left[f_{mn}\left(\left\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\dots,d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}}\geq\varepsilon\right\}\right|$$

and

$$H_{rs} = \left|\left\{m,n\in I_{rs}:\left[f_{mn}\left(\left\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\dots,d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}}<\varepsilon\right\}\right|.$$

then we have

$$\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left[f_{mn}\left(\left\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\dots,d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}} = \sum_{m\in G_{rs}}\sum_{n\in G_{rs}}\left[f_{mn}\left(\left\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\dots,d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}} + \sum_{m\in H_{rs}}\sum_{n\in H_{rs}}\left[f_{mn}\left(\left\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\dots,d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}} \leq \max(T^h,T^H)G_{rs}+\max(\varepsilon^h,\varepsilon^H).$$

Taking the limit as $\varepsilon\to0$ and $r,s\to\infty$, it follows that $\Gamma^2(X)\in(\breve{w}_\lambda^F,q)$.

Proof (c): Follows from (a) and (b).

Theorem 3.3: If $\liminf_{rs}\left(\frac{\lambda_{rs}}{rs}\right)>0$, then $\breve{S}^{2F}\subset\breve{S}_\lambda^{2F}$.
Proof: Let $\Gamma^2(X)\in\breve{S}^{2F}$. For given $\varepsilon>0$, we get

$$\left|\left\{m\leq r,n\leq s:\left[f_{mn}\left(\left\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\dots,d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}}\geq\varepsilon\right\}\right|\supset G_{rs}$$

where G_{rs} is in the Theorem of 3.2 (b). Thus,

$$\left|\left\{m\leq r,n\leq s:\left[f_{mn}\left(\left\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\dots,d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}}\geq\varepsilon\right\}\right|\geq G_{rs}=\frac{\lambda_{rs}}{rs}.$$

Taking limit as $r,s\to\infty$ and using $\liminf_{rs}\left(\frac{\lambda_{rs}}{rs}\right)>0$, we get $\Gamma^2(X)\in\breve{S}_\lambda^{2F}$.

Theorem 3.4: Let $0<u_{mn}\leq v_{mn}$ and $(u_{mn}v_{mn}^{-1})$ be double analytic. Then $(\breve{w}_\lambda^{2F},v)\subset(\breve{w}_\lambda^{2F},u)$.

Proof: Let $\Gamma^2(X)\in(\breve{w}_\lambda^{2F},v)$. Let

$$w_{mn} = \left[f_{mn}\left(\left\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\dots,d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}}$$

for all $r, s \in \mathbb{N}$ and $\lambda_{mn} = u_{mn} v_{mn}^{-1}$ for all $m, n \in \mathbb{N}$. Then $0 < \lambda_{mn} \leq 1$ for all $m, n \in \mathbb{N}$. Let b be a constant such that $0 < b \leq \lambda_{mn} \leq 1$ for all $m, n \in \mathbb{N}$.

Define the sequences (k_{mn}) and (ℓ_{mn}) as follows: For $w_{mn} \geq 1$, let $(k_{mn}) = (w_{mn})$ and $\ell_{mn} = 0$ and for $w_{mn} < 1$, let $k_{mn} = 0$ and $\ell_{mn} = w_{mn}$. Then it is clear that for all $m, n \in \mathbb{N}$, we have $w_{mn} = k_{mn} + \ell_{mn}$ and $w_{mn}^{\lambda_{mn}} = k_{mn}^{\lambda_{mn}} + \ell_{mn}^{\lambda_{mn}}$. Now it follows that $k_{mn}^{\lambda_{mn}} \leq k_{mn} \leq w_{mn}$ and $\ell_{mn}^{\lambda_{mn}} \leq \ell_{mn}^{\lambda}$. Therefore,

$$\sum_{m\in I_{rs}} \sum_{n\in I_{rs}} \left[f_{mn}\left(\left\|\mu_{mn}(w_{mn}^{\lambda_{mn}}), (d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}} =$$

$$\sum_{m\in I_{rs}} \sum_{n\in I_{rs}} \left[f_{mn}\left(\left\|\mu_{mn}(k_{mn} + \ell_{mn})^{\lambda_{mn}}, (d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}} =$$
$$\sum_{m\in I_{rs}} \sum_{n\in I_{rs}} \left[f_{mn}\left(\left\|\mu_{mn}(w_{mn}), (d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}}$$
$$+ \sum_{m\in I_{rs}} \sum_{n\in I_{rs}} \left[f_{mn}\left(\left\|\mu_{mn}(\ell_{mn})^{\lambda_{mn}}, (d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}}$$

now for each r, s,

$$\sum_{m\in I_{rs}} \sum_{n\in I_{rs}} \left[f_{mn}\left(\left\|\mu_{mn}(\ell_{mn})^{\lambda}, (d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}} =$$
$$\sum_{m\in I_{rs}} \sum_{n\in I_{rs}} \left[f_{mn}\left(\left\|\mu_{mn}\left((\ell_{mn})^{\lambda}\left(\frac{1}{\lambda_{rs}}\right)^{1-\lambda}\right), (d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}} \leq$$
$$\left[\sum_{m\in I_{rs}} \sum_{n\in I_{rs}} \left[f_{mn} \| \left(\left(\mu_{mn}((\ell_{mn})^{\lambda})^{\lambda}\right)^{1/\lambda}, (d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0)) \|_p\right)\right]^{q_{mn}}\right]^{\lambda}$$

Theorem 3.5: $\breve{\Lambda}^{2F} = \breve{w}^{2F}_{\lambda,\Lambda^2}$, where

$$\breve{w}^{2F}_{\lambda,\Lambda^2} = X = (X_{mn}): \sup_{rs} \sum_{m\in I_{rs}} \sum_{n\in I_{rs}} \left[f_{mn}\left(\left\|\mu_{mn}(x), (d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}} < \infty$$

Proof: Let $X = (X_{mn}) \in \breve{w}^{2F}_{\lambda,\Lambda^2}$. Then there exists a constant $T_1 > 0$ such that,

$$\left[f_{mn}\left(\left\|\mu_{mn}(x), (d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}} \leq \sup_{rs} \sum_{m\in I_{rs}} \sum_{n\in I_{rs}} \left[f_{mn}\left(\left\|\mu_{mn}(x), (d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}} \leq T_1$$

for all $r, s \in \mathbb{N}$. Therefore we have $X = (X_{mn}) \in \breve{\Lambda}^{2F}$. Conversely, let $X = (X_{mn}) \in \breve{\Lambda}^{2F}$. Then there exists a constant $T_2 > 0$ such that,

$$\left[f_{mn}\left(\left\|\mu_{mn}(x), (d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}} \leq T_2$$

for all m, n and r, s. So,

$$\sum_{m\in I_{rs}} \sum_{n\in I_{rs}} \left[f_{mn}\left(\left\|\mu_{mn}(x), (d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\right\|_p\right)\right]^{q_{mn}} \leq T_2 \sum_{m\in I_{rs}} \sum_{n\in I_{rs}} 1 \leq T_2,$$

for all m, n and r, s. Thus $X = (X_{mn}) \in \breve{w}^{2F}_{\lambda,\Lambda^2}$.

4. Conclusion

The statistical approach implement of function space of $\mathbf{\Gamma^2}$ and then well defined of definitions. This logic hypothesis of statistical new approaches of testing and verification of prove the results. This is vital role of that research paper.

Acknowledgement

The authors are extremely grateful to the anonymous learned referee(s) for their keen reading, valuable suggestion and constructive comments for the improvement of the manuscript. The authors are thankful to the editor(s) and reviewers of Applied Mathematics and Information Sciences and also authors wish to thank the Department of Science and Technology, Government of India for the financial sanction towards this work under FIST program SR/FST/MSI-107/2015.

Competing interests

The authors declare that there is no conflict of interests regarding the publication of this research paper.

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